Further Insights and Challenges

63. An object is tossed into the air vertically from ground level with initial velocity v_0 ft/s at time $t = 0$. Find the average speed of the object over the time interval $[0, T]$, where T is the time the object returns to earth.

SOLUTION The height is given by $h(t) = v_0 t - 16t^2$. The ball is at ground level at time $t = 0$ and $T = v_0/16$. The velocity is given by $v(t) = v_0 - 32t$ and thus the speed is given by $s(t) = |v_0 - 32t|$. The average speed is

$$\frac{1}{v_0/16 - 0}\int_0^{v_0/16} |v_0 - 32t|\, dt = \frac{16}{v_0}\int_0^{v_0/32} (v_0 - 32t)\, dt + \frac{16}{v_0}\int_{v_0/32}^{v_0/16} (32t - v_0)\, dt$$
$$= \frac{16}{v_0}\left(v_0 t - 16t^2\right)\Big|_0^{v_0/32} + \frac{16}{v_0}\left(16t^2 - v_0 t\right)\Big|_{v_0/32}^{v_0/16} = v_0/2.$$

64. Review the MVT stated in Section 4.3 (Theorem 1, p. 226) and show how it can be used, together with the Fundamental Theorem of Calculus, to prove the MVT for Integrals.

SOLUTION The Mean Value Theorem essentially states that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some $c \in (a, b)$. Let F be any antiderivative of f. Then

$$f(c) = F'(c) = \frac{F(b) - F(a)}{b - a} = \frac{1}{b - a}\,(F(b) - F(a)) = \frac{1}{b - a}\int_a^b f(x)\, dx.$$

6.3 Volumes of Revolution

Preliminary Questions

1. Which of the following is a solid of revolution?

(a) Sphere **(b)** Pyramid **(c)** Cylinder **(d)** Cube

SOLUTION The sphere and the cylinder have circular cross sections; hence, these are solids of revolution. The pyramid and cube do not have circular cross sections, so these are not solids of revolution.

2. True or false? When the region under a single graph is rotated about the x-axis, the cross sections of the solid perpendicular to the x-axis are circular disks.

SOLUTION True. The cross sections will be disks with radius equal to the value of the function.

3. True or false? When the region between two graphs is rotated about the x-axis, the cross sections to the solid perpendicular to the x-axis are circular disks.

SOLUTION False. The cross sections may be washers.

4. Which of the following integrals expresses the volume obtained by rotating the area between $y = f(x)$ and $y = g(x)$ over $[a, b]$ around the x-axis? [Assume $f(x) \geq g(x) \geq 0$.]

(a) $\pi \int_a^b \left(f(x) - g(x)\right)^2 dx$

(b) $\pi \int_a^b \left(f(x)^2 - g(x)^2\right) dx$

SOLUTION The correct answer is **(b)**. Cross sections of the solid will be washers with outer radius $f(x)$ and inner radius $g(x)$. The area of the washer is then $\pi f(x)^2 - \pi g(x)^2 = \pi(f(x)^2 - g(x)^2)$.

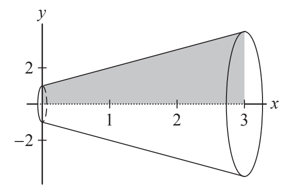
Exercises

In Exercises 1–4, (a) sketch the solid obtained by revolving the region under the graph of $f(x)$ about the x-axis over the given interval, (b) describe the cross section perpendicular to the x-axis located at x, and (c) calculate the volume of the solid.

1. $f(x) = x + 1$, $[0, 3]$

SOLUTION

(a) A sketch of the solid of revolution is shown below:

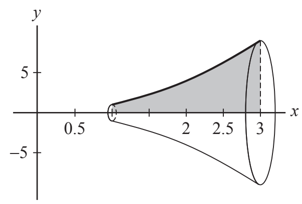


(b) Each cross section is a disk with radius $x + 1$.
(c) The volume of the solid of revolution is

$$\pi \int_0^3 (x+1)^2\,dx = \pi \int_0^3 (x^2 + 2x + 1)\,dx = \pi \left(\frac{1}{3}x^3 + x^2 + x \right)\Bigg|_0^3 = 21\pi.$$

2. $f(x) = x^2, \quad [1, 3]$

SOLUTION
(a) A sketch of the solid of revolution is shown below:

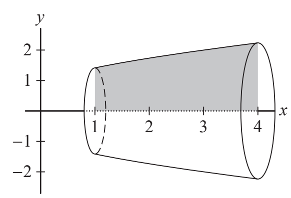


(b) Each cross section is a disk of radius x^2.
(c) The volume of the solid of revolution is

$$\pi \int_1^3 \left(x^2\right)^2\,dx = \pi \left(\frac{x^5}{5} \right)\Bigg|_1^3 = \frac{242\pi}{5}.$$

3. $f(x) = \sqrt{x+1}, \quad [1, 4]$

SOLUTION
(a) A sketch of the solid of revolution is shown below:

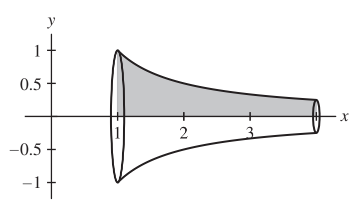


(b) Each cross section is a disk with radius $\sqrt{x+1}$.
(c) The volume of the solid of revolution is

$$\pi \int_1^4 (\sqrt{x+1})^2\,dx = \pi \int_1^4 (x+1)\,dx = \pi \left(\frac{1}{2}x^2 + x \right)\Bigg|_1^4 = \frac{21\pi}{2}.$$

4. $f(x) = x^{-1}, \quad [1, 4]$

SOLUTION
(a) A sketch of the solid of revolution is shown below:



(b) Each cross section is a disk with radius x^{-1}.
(c) The volume of the solid of revolution is

$$\pi \int_1^4 (x^{-1})^2\,dx = \pi \int_1^4 x^{-2}\,dx = \pi\,(-x)^{-1}\Big|_1^4 = \frac{3\pi}{4}.$$

In Exercises 5–12, find the volume of revolution about the x-axis for the given function and interval.

5. $f(x) = x^2 - 3x, \quad [0, 3]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_0^3 (x^2 - 3x)^2\,dx = \pi \int_0^3 (x^4 - 6x^3 + 9x^2)\,dx = \pi \left(\frac{1}{5}x^5 - \frac{3}{2}x^4 + 3x^3 \right)\Big|_0^3 = \frac{81\pi}{10}.$$

6. $f(x) = \frac{1}{x^2}, \quad [1, 4]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_1^4 (x^{-2})^2\,dx = \pi \int_1^4 x^{-4}\,dx = \pi \left(-\frac{1}{3}x^{-3} \right)\Big|_1^4 = \frac{21\pi}{64}.$$

7. $f(x) = x^{5/3}, \quad [1, 8]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_1^8 (x^{5/3})^2\,dx = \pi \int_1^8 x^{10/3}\,dx = \frac{3\pi}{13} x^{13/3}\Big|_1^8 = \frac{3\pi}{13}(2^{13} - 1) = \frac{24573\pi}{13}.$$

8. $f(x) = 4 - x^2, \quad [0, 2]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_0^2 (4 - x^2)^2\,dx = \pi \int_0^2 (16 - 8x^2 + x^4)\,dx = \pi \left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right)\Big|_0^2 = \frac{256\pi}{15}.$$

9. $f(x) = \frac{2}{x+1}, \quad [1, 3]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_1^3 \left(\frac{2}{x+1} \right)^2 dx = 4\pi \int_1^3 (x+1)^{-2}\,dx = -4\pi\,(x+1)^{-1}\Big|_1^3 = \pi.$$

10. $f(x) = \sqrt{x^4 + 1}, \quad [1, 3]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_1^3 (\sqrt{x^4 + 1})^2\,dx = \pi \int_1^3 (x^4 + 1)\,dx = \pi \left(\frac{1}{5}x^5 + x \right)\Big|_1^3 = \frac{252\pi}{5}.$$

11. $f(x) = e^x, \quad [0, 1]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_0^1 (e^x)^2\,dx = \frac{1}{2}\pi e^{2x}\Big|_0^1 = \frac{1}{2}\pi(e^2 - 1).$$

12. $f(x) = \sqrt{\cos x \sin x}, \quad \left[0, \frac{\pi}{2}\right]$

SOLUTION The volume of the solid of revolution is

$$\pi \int_0^{\pi/2} (\sqrt{\cos x \sin x})^2\,dx = \pi \int_0^{\pi/2} (\cos x \sin x)\,dx = \frac{\pi}{2} \int_0^{\pi/2} \sin 2x\,dx = \frac{\pi}{4}\,(-\cos 2x)\Big|_0^{\pi/2} = \frac{\pi}{2}.$$

In Exercises 13 and 14, R is the shaded region in Figure 1.

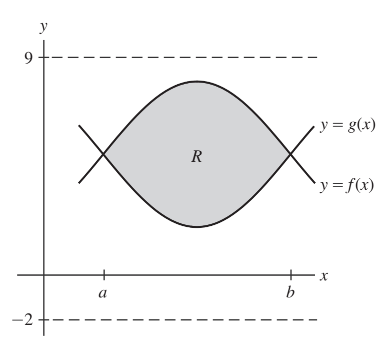


FIGURE 1

13. Which of the integrands (i)–(iv) is used to compute the volume obtained by rotating region R about $y = -2$?
(i) $(f(x)^2 + 2^2) - (g(x)^2 + 2^2)$
(ii) $(f(x) + 2)^2 - (g(x) + 2)^2$
(iii) $(f(x)^2 - 2^2) - (g(x)^2 - 2^2)$
(iv) $(f(x) - 2)^2 - (g(x) - 2)^2$

SOLUTION when the region R is rotated about $y = -2$, the outer radius is $f(x) - (-2) = f(x) + 2$ and the inner radius is $g(x) - (-2) = g(x) + 2$. Thus, the appropriate integrand is **(ii)**: $(f(x) + 2)^2 - (g(x) + 2)^2$.

14. Which of the integrands (i)–(iv) is used to compute the volume obtained by rotating R about $y = 9$?
(i) $(9 + f(x))^2 - (9 + g(x))^2$
(ii) $(9 + g(x))^2 - (9 + f(x))^2$
(iii) $(9 - f(x))^2 - (9 - g(x))^2$
(iv) $(9 - g(x))^2 - (9 - f(x))^2$

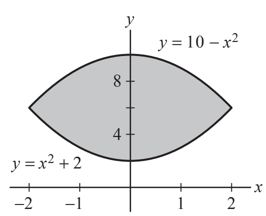
SOLUTION when the region R is rotated about $y = 9$, the outer radius is $9 - g(x)$ and the inner radius is $9 - f(x)$. Thus, the appropriate integrand is **(iv)**: $(9 - g(x))^2 - (9 - f(x))^2$.

In Exercises 15–20, (a) sketch the region enclosed by the curves, (b) describe the cross section perpendicular to the x-axis located at x, and (c) find the volume of the solid obtained by rotating the region about the x-axis.

15. $y = x^2 + 2, \quad y = 10 - x^2$

SOLUTION

(a) Setting $x^2 + 2 = 10 - x^2$ yields $2x^2 = 8$, or $x^2 = 4$. The two curves therefore intersect at $x = \pm 2$. The region enclosed by the two curves is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a washer with outer radius $R = 10 - x^2$ and inner radius $r = x^2 + 2$.

(c) The volume of the solid of revolution is

$$\pi \int_{-2}^{2} \left((10 - x^2)^2 - (x^2 + 2)^2 \right) dx = \pi \int_{-2}^{2} (96 - 24x^2)\, dx = \pi \left(96x - 8x^3 \right) \Big|_{-2}^{2} = 256\pi.$$

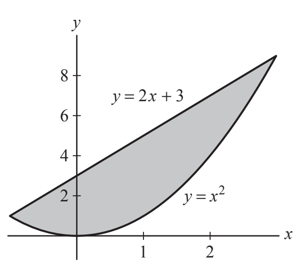
16. $y = x^2, \quad y = 2x + 3$

SOLUTION

(a) Setting $x^2 = 2x + 3$ yields

$$0 = x^2 - 2x - 3 = (x - 3)(x + 1).$$

The two curves therefore intersect at $x = -1$ and $x = 3$. The region enclosed by the two curves is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a washer with outer radius $R = 2x + 3$ and inner radius $r = x^2$.

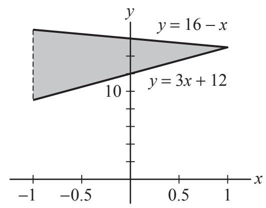
(c) The volume of the solid of revolution is

$$\pi \int_{-1}^{3} \left((2x + 3)^2 - (x^2)^2 \right) dx = \pi \int_{-1}^{3} (4x^2 + 12x + 9 - x^4)\, dx = \pi \left(\frac{4}{3}x^3 + 6x^2 + 9x - \frac{1}{5}x^5 \right) \Big|_{-1}^{3} = \frac{1088\pi}{15}.$$

17. $y = 16 - x, \quad y = 3x + 12, \quad x = -1$

SOLUTION

(a) Setting $16 - x = 3x + 12$, we find that the two lines intersect at $x = 1$. The region enclosed by the two curves is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a washer with outer radius $R = 16 - x$ and inner radius $r = 3x + 12$.

(c) The volume of the solid of revolution is

$$\pi \int_{-1}^{1} \left((16-x)^2 - (3x+12)^2 \right) dx = \pi \int_{-1}^{1} (112 - 104x - 8x^2)\, dx = \pi \left(112x - 52x^2 - \frac{8}{3}x^3 \right)\Big|_{-1}^{1} = \frac{656\pi}{3}.$$

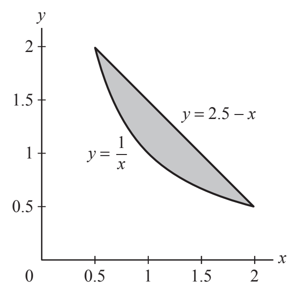
18. $y = \dfrac{1}{x}, \quad y = \dfrac{5}{2} - x$

SOLUTION

(a) Setting $\frac{1}{x} = \frac{5}{2} - x$ yields

$$0 = x^2 - \frac{5}{2}x + 1 = (x-2)\left(x - \frac{1}{2}\right).$$

The two curves therefore intersect at $x = 2$ and $x = \frac{1}{2}$. The region enclosed by the two curves is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a washer with outer radius $R = \frac{5}{2} - x$ and inner radius $r = x^{-1}$.

(c) The volume of the solid of revolution is

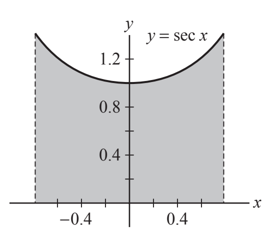
$$\pi \int_{1/2}^{2} \left(\left(\frac{5}{2} - x\right)^2 - \left(\frac{1}{x}\right)^2 \right) dx = \pi \int_{1/2}^{2} \left(\frac{25}{4} - 5x + x^2 - x^{-2} \right) dx$$

$$= \pi \left(\frac{25}{4}x - \frac{5}{2}x^2 + \frac{1}{3}x^3 + x^{-1} \right)\Big|_{1/2}^{2} = \frac{9\pi}{8}.$$

19. $y = \sec x, \quad y = 0, \quad x = -\dfrac{\pi}{4}, \quad x = \dfrac{\pi}{4}$

SOLUTION

(a) The region in question is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a circular disk with radius $R = \sec x$.

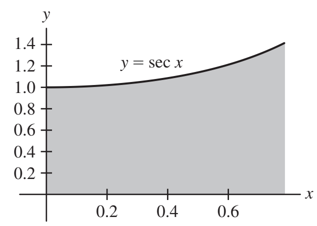
(c) The volume of the solid of revolution is

$$\pi \int_{-\pi/4}^{\pi/4} (\sec x)^2\, dx = \pi\,(\tan x)\Big|_{-\pi/4}^{\pi/4} = 2\pi.$$

20. $y = \sec x, \quad y = 0, \quad x = 0, \quad x = \dfrac{\pi}{4}$

SOLUTION

(a) The region in question is shown in the figure below.



(b) When the region is rotated about the x-axis, each cross section is a circular disk with radius $R = \sec x$.

(c) The volume of the solid of revolution is

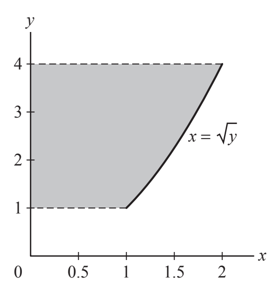
$$\pi \int_{0}^{\pi/4} (\sec x)^2\, dx = \pi\,(\tan x)\Big|_{0}^{\pi/4} = \pi.$$

In Exercises 21–24, find the volume of the solid obtained by rotating the region enclosed by the graphs about the y-axis over the given interval.

21. $x = \sqrt{y}, \quad x = 0; \quad 1 \le y \le 4$

SOLUTION When the region in question (shown in the figure below) is rotated about the y-axis, each cross section is a disk with radius $\sqrt{y}$. The volume of the solid of revolution is

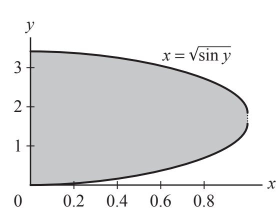
$$\pi \int_1^4 \left(\sqrt{y}\right)^2\, dy = \frac{\pi y^2}{2}\Big|_1^4 = \frac{15\pi}{2}.$$



22. $x = \sqrt{\sin y}, \quad x = 0; \quad 0 \le y \le \pi$

SOLUTION When the region in question (shown in the figure below) is rotated about the y-axis, each cross section is a disk with radius $\sqrt{\sin y}$. The volume of the solid of revolution is

$$\pi \int_0^{\pi} \left(\sqrt{\sin y}\right)^2\, dy = \pi\,(-\cos y)\Big|_0^{\pi} = 2\pi.$$



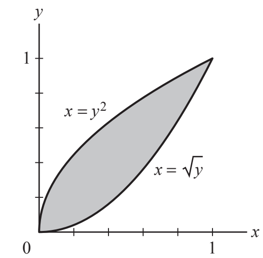
23. $x = y^2, \quad x = \sqrt{y}$

SOLUTION Setting $y^2 = \sqrt{y}$ and then squaring both sides yields

$$y^4 = y \quad \text{or} \quad y^4 - y = y(y^3 - 1) = 0,$$

so the two curves intersect at $y = 0$ and $y = 1$. When the region in question (shown in the figure below) is rotated about the y-axis, each cross section is a washer with outer radius $R = \sqrt{y}$ and inner radius $r = y^2$. The volume of the solid of revolution is

$$\pi \int_0^1 \left((\sqrt{y})^2 - (y^2)^2\right) dy = \pi \left(\frac{y^2}{2} - \frac{y^5}{5}\right)\Bigg|_0^1 = \frac{3\pi}{10}.$$



24. $x = 4 - y, \quad x = 16 - y^2$

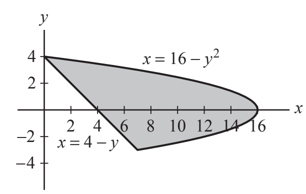
SOLUTION Setting $4 - y = 16 - y^2$ yields

$$0 = y^2 - y - 12 = (y - 4)(y + 3),$$

so the two curves intersect at $y = -3$ and $y = 4$. When the region enclosed by the two curves (shown in the figure below) is rotated about the y-axis, each cross section is a washer with outer radius $R = 16 - y^2$ and inner radius $r = 4 - y$. The volume of the solid of revolution is

$$\pi \int_{-3}^4 \left((16 - y^2)^2 - (4 - y)^2\right) dy = \pi \int_{-3}^4 \left(y^4 - 33y^2 + 8y + 240\right) dy$$

$$= \pi \left(\frac{1}{5}y^5 - 11y^3 + 4y^2 + 240y\right)\Bigg|_{-3}^4 = \frac{4802\pi}{5}.$$



25. Rotation of the region in Figure 2 about the y-axis produces a solid with two types of different cross sections. Compute the volume as a sum of two integrals, one for $-12 \le y \le 4$ and one for $4 \le y \le 12$.

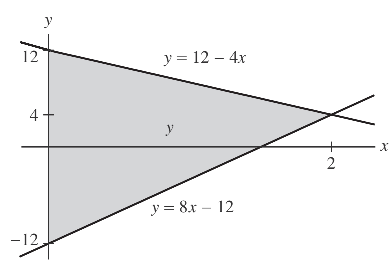


FIGURE 2

SOLUTION For $-12 \le y \le 4$, the cross section is a disk with radius $\frac{1}{8}(y + 12)$; for $4 \le y \le 12$, the cross section is a disk with radius $\frac{1}{4}(12 - y)$. Therefore, the volume of the solid of revolution is

$$V = \frac{\pi}{8} \int_{-12}^4 (y + 12)^2 \, dy + \frac{\pi}{4} \int_4^{12} (12 - y)^2 \, dy$$

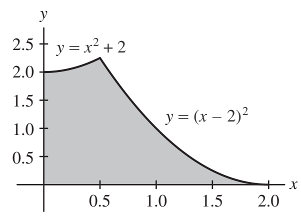
$$= \frac{\pi}{24}(y + 12)^3 \Big|_{-12}^4 - \frac{\pi}{12}(12 - y)^3 \Big|_4^{12}$$

$$= \frac{512\pi}{3} + \frac{128\pi}{3} = \frac{640\pi}{3}.$$

26. Let R be the region enclosed by $y = x^2 + 2$, $y = (x-2)^2$ and the axes $x = 0$ and $y = 0$. Compute the volume V obtained by rotating R about the x-axis. *Hint:* Express V as a sum of two integrals.

SOLUTION Setting $x^2 + 2 = (x-2)^2$ yields $4x = 2$ or $x = 1/2$. When the region enclosed by the two curves and the coordinate axes (shown in the figure below) is rotated about the x-axis, there are two different cross sections. For $0 \le x \le 1/2$, the cross section is a disk of radius $x^2 + 2$; for $1/2 \le x \le 2$, the cross section is a disk of radius $(x-2)^2$. The volume of the solid of revolution is therefore

$$V = \pi \int_0^{1/2} (x^2+2)\,dx + \pi \int_{1/2}^{2} (x-2)^2\,dx$$
$$= \pi\left(\frac{1}{3}x^3 + 2x\right)\Big|_0^{1/2} + \frac{\pi}{3}(x-2)^3\Big|_{1/2}^{2}$$
$$= \frac{25\pi}{24} + \frac{9\pi}{8} = \frac{13\pi}{6}.$$



In Exercises 27–32, find the volume of the solid obtained by rotating region A in Figure 3 about the given axis.

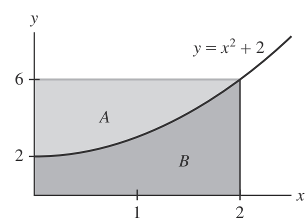


FIGURE 3

27. x-axis

SOLUTION Rotating region A about the x-axis produces a solid whose cross sections are washers with outer radius $R = 6$ and inner radius $r = x^2 + 2$. The volume of the solid of revolution is

$$\pi \int_0^2 \left((6)^2 - (x^2+2)^2\right) dx = \pi \int_0^2 (32 - 4x^2 - x^4)\,dx = \pi\left(32x - \frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2 = \frac{704\pi}{15}.$$

28. $y = -2$

SOLUTION Rotating region A about $y = -2$ produces a solid whose cross sections are washers with outer radius $R = 6 - (-2) = 8$ and inner radius $r = x^2 + 2 - (-2) = x^2 + 4$. The volume of the solid of revolution is

$$\pi \int_0^2 \left((8)^2 - (x^2+4)^2\right) dx = \pi \int_0^2 (48 - 8x^2 - x^4)\,dx = \pi\left(48x - \frac{8}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2 = \frac{1024\pi}{15}.$$

29. $y = 2$

SOLUTION Rotating the region A about $y = 2$ produces a solid whose cross sections are washers with outer radius $R = 6 - 2 = 4$ and inner radius $r = x^2 + 2 - 2 = x^2$. The volume of the solid of revolution is

$$\pi \int_0^2 \left(4^2 - (x^2)^2\right) dx = \pi\left(16x - \frac{1}{5}x^5\right)\Big|_0^2 = \frac{128\pi}{5}.$$

30. y-axis

SOLUTION Rotating region A about the y-axis produces a solid whose cross sections are disks with radius $R = \sqrt{y-2}$. Note that here we need to integrate along the y-axis. The volume of the solid of revolution is

$$\pi \int_2^6 (\sqrt{y-2})^2\,dy = \pi \int_2^6 (y-2)\,dy = \pi\left(\frac{1}{2}y^2 - 2y\right)\Big|_2^6 = 8\pi.$$

31. $x = -3$

SOLUTION Rotating region A about $x = -3$ produces a solid whose cross sections are washers with outer radius $R = \sqrt{y-2} - (-3) = \sqrt{y-2} + 3$ and inner radius $r = 0 - (-3) = 3$. The volume of the solid of revolution is

$$\pi \int_2^6 \left((3 + \sqrt{y-2})^2 - (3)^2 \right) dy = \pi \int_2^6 (6\sqrt{y-2} + y - 2)\, dy = \pi \left(4(y-2)^{3/2} + \frac{1}{2}y^2 - 2y \right) \Big|_2^6 = 40\pi.$$

32. $x = 2$

SOLUTION Rotating region A about $x = 2$ produces a solid whose cross sections are washers with outer radius $R = 2 - 0 = 2$ and inner radius $r = 2 - \sqrt{y-2}$. The volume of the solid of revolution is

$$\pi \int_2^6 \left(2^2 - (2 - \sqrt{y-2})^2 \right) dy = \pi \int_2^6 \left(4\sqrt{y-2} - y + 2 \right) dy = \pi \left(\frac{8}{3}(y-2)^{3/2} - \frac{1}{2}y^2 + 2y \right) \Big|_2^6 = \frac{40\pi}{3}.$$

In Exercises 33–38, find the volume of the solid obtained by rotating region B in Figure 3 about the given axis.

33. x-axis

SOLUTION Rotating region B about the x-axis produces a solid whose cross sections are disks with radius $R = x^2 + 2$. The volume of the solid of revolution is

$$\pi \int_0^2 (x^2+2)^2\, dx = \pi \int_0^2 (x^4 + 4x^2 + 4)\, dx = \pi \left(\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x \right) \Big|_0^2 = \frac{376\pi}{15}.$$

34. $y = -2$

SOLUTION Rotating region B about $y = -2$ produces a solid whose cross sections are washers with outer radius $R = x^2 + 2 - (-2) = x^2 + 4$ and inner radius $r = 0 - (-2) = 2$. The volume of the solid of revolution is

$$\pi \int_0^2 \left((x^2+4)^2 - (2)^2 \right) dx = \pi \int_0^2 (x^4 + 8x^2 + 12)\, dx = \pi \left(\frac{1}{5}x^5 + \frac{8}{3}x^3 + 12x \right) \Big|_0^2 = \frac{776\pi}{15}.$$

35. $y = 6$

SOLUTION Rotating region B about $y = 6$ produces a solid whose cross sections are washers with outer radius $R = 6 - 0 = 6$ and inner radius $r = 6 - (x^2 + 2) = 4 - x^2$. The volume of the solid of revolution is

$$\pi \int_0^2 \left(6^2 - (4 - x^2)^2 \right) dy = \pi \int_0^2 \left(20 + 8x^2 - x^4 \right) dy = \pi \left(20x + \frac{8}{3}x^3 - \frac{1}{5}x^5 \right) \Big|_0^2 = \frac{824\pi}{15}.$$

36. y-axis

Hint for Exercise 36: Express the volume as a sum of two integrals along the y-axis or use Exercise 30.

SOLUTION Rotating region B about the y-axis produces a solid with two different cross sections. For each $y \in [0, 2]$, the cross section is a disk with radius $R = 2$; for each $y \in [2, 6]$, the cross section is a washer with outer radius $R = 2$ and inner radius $r = \sqrt{y-2}$. The volume of the solid of revolution is

$$\pi \int_0^2 (2)^2\, dy + \pi \int_2^6 \left((2)^2 - (\sqrt{y-2})^2 \right) dy = \pi \int_0^2 4\, dy + \pi \int_2^6 (6 - y)\, dy$$
$$= \pi\, (4y) \Big|_0^2 + \pi \left(6y - \frac{1}{2}y^2 \right) \Big|_2^6 = 16\pi.$$

Alternately, we recognize that rotating both region A and region B about the y-axis produces a cylinder of radius $R = 2$ and height $h = 6$. The volume of this cylinder is $\pi(2)^2 \cdot 6 = 24\pi$. In Exercise 30, we found that the volume of the solid generated by rotating region A about the y-axis to be 8π. Therefore, the volume of the solid generated by rotating region B about the y-axis is $24\pi - 8\pi = 16\pi$.

37. $x = 2$

SOLUTION Rotating region B about $x = 2$ produces a solid with two different cross sections. For each $y \in [0, 2]$, the cross section is a disk with radius $R = 2$; for each $y \in [2, 6]$, the cross section is a disk with radius $R = 2 - \sqrt{y-2}$. The volume of the solid of revolution is

$$\pi \int_0^2 (2)^2\, dy + \pi \int_2^6 (2 - \sqrt{y-2})^2\, dy = \pi \int_0^2 4\, dy + \pi \int_2^6 (2 + y - 4\sqrt{y-2})\, dy$$
$$= \pi\, (4y) \Big|_0^2 + \pi \left(2y + \frac{1}{2}y^2 - \frac{8}{3}(y-2)^{3/2} \right) \Big|_2^6 = \frac{32\pi}{3}.$$

38. $x = -3$

SOLUTION Rotating region B about $x = -3$ produces a solid with two different cross sections. For each $y \in [0, 2]$, the cross section is a washer with outer radius $R = 2 - (-3) = 5$ and inner radius $r = 0 - (-3) = 3$; for each $y \in [2, 6]$, the cross section is a washer with outer radius $R = 2 - (-3) = 5$ and inner radius $r = \sqrt{y-2} - (-3) = \sqrt{y-2} + 3$. The volume of the solid of revolution is

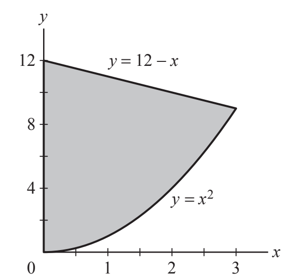
$$\pi \int_0^2 \left((5)^2 - (3)^2\right) dy + \pi \int_2^6 \left((5)^2 - (\sqrt{y-2} + 3)^2\right) dy$$
$$= \pi \int_0^2 16\, dy + \pi \int_2^6 (18 - y - 6\sqrt{y-2})\, dy$$
$$= \pi\, (16y)\Big|_0^2 + \pi \left(18y - \frac{1}{2}y^2 - 4(y-2)^{3/2}\right)\Big|_2^6 = 56\pi.$$

In Exercises 39–52, find the volume of the solid obtained by rotating the region enclosed by the graphs about the given axis.

39. $y = x^2, \quad y = 12 - x, \quad x = 0, \quad$ about $y = -2$

SOLUTION Rotating the region enclosed by $y = x^2$, $y = 12 - x$ and the y-axis (shown in the figure below) about $y = -2$ produces a solid whose cross sections are washers with outer radius $R = 12 - x - (-2) = 14 - x$ and inner radius $r = x^2 - (-2) = x^2 + 2$. The volume of the solid of revolution is

$$\pi \int_0^3 \left((14 - x)^2 - (x^2 + 2)^2\right) dx = \pi \int_0^3 (192 - 28x - 3x^2 - x^4)\, dx$$
$$= \pi \left(192x - 14x^2 - x^3 - \frac{1}{5}x^5\right)\Big|_0^3 = \frac{1872\pi}{5}.$$



40. $y = x^2, \quad y = 12 - x, \quad x = 0, \quad$ about $y = 15$

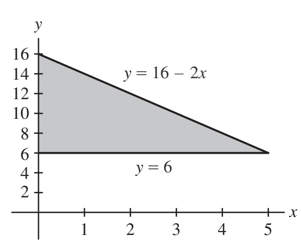
SOLUTION Rotating the region enclosed by $y = x^2$, $y = 12 - x$ and the y-axis (see the figure in the previous exercise) about $y = 15$ produces a solid whose cross sections are washers with outer radius $R = 15 - x^2$ and inner radius $r = 15 - (12 - x) = 3 + x$. The volume of the solid of revolution is

$$\pi \int_0^3 \left((15 - x^2)^2 - (3 + x)^2\right) dx = \pi \int_0^3 (216 - 6x - 31x^2 + x^4)\, dx$$
$$= \pi \left(216x - 3x^2 - \frac{31}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^3 = \frac{1953\pi}{5}.$$

41. $y = 16 - 2x, \quad y = 6, \quad x = 0, \quad$ about x-axis

SOLUTION Rotating the region enclosed by $y = 16 - 2x$, $y = 6$ and the y-axis (shown in the figure below) about the x-axis produces a solid whose cross sections are washers with outer radius $R = 16 - 2x$ and inner radius $r = 6$. The volume of the solid of revolution is

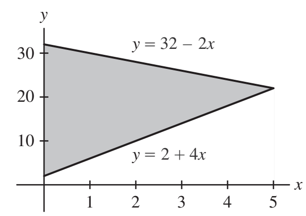
$$\pi \int_0^5 \left((16 - 2x)^2 - 6^2\right) dx = \pi \int_0^5 (220 - 64x + 4x^2)\, dx$$
$$= \pi \left(220x - 32x^2 + \frac{4}{3}x^3\right)\Big|_0^5 = \frac{1400\pi}{3}.$$



42. $y = 32 - 2x, \quad y = 2 + 4x, \quad x = 0,$ about y-axis

SOLUTION Rotating the region enclosed by $y = 32 - 2x$, $y = 2 + 4x$ and the y-axis (shown in the figure below) about the y-axis produces a solid with two different cross sections. For $2 \le y \le 22$, the cross section is a disk of radius $\frac{1}{4}(y - 2)$; for $22 \le y \le 32$, the cross section is a disk of radius $\frac{1}{2}(32 - y)$. The volume of the solid of revolution is

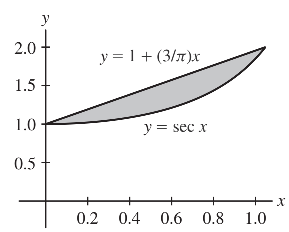
$$V = \frac{\pi}{4}\int_2^{22} (y-2)^2\, dy + \frac{\pi}{2}\int_{22}^{32} (32-y)^2\, dy$$
$$= \frac{\pi}{12}(y-2)^3\Big|_2^{22} - \frac{\pi}{6}(32-y)^3\Big|_{22}^{32}$$
$$= \frac{2000\pi}{3} + \frac{500\pi}{3} = \frac{2500\pi}{3}.$$



43. $y = \sec x, \quad y = 1 + \frac{3}{\pi}x,$ about x-axis

SOLUTION We first note that $y = \sec x$ and $y = 1 + (3/\pi)x$ intersect at $x = 0$ and $x = \pi/3$. Rotating the region enclosed by $y = \sec x$ and $y = 1 + (3/\pi)x$ (shown in the figure below) about the x-axis produces a cross section that is a washer with outer radius $R = 1 + (3/\pi)x$ and inner radius $r = \sec x$. The volume of the solid of revolution is

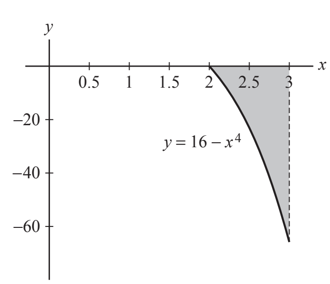
$$V = \pi\int_0^{\pi/3} \left(\left(1 + \frac{3}{\pi}x\right)^2 - \sec^2 x\right) dx$$
$$= \pi\int_0^{\pi/3} \left(1 + \frac{6}{\pi}x + \frac{9}{\pi^2}x^2 - \sec^2 x\right) dx$$
$$= \pi\left(x + \frac{3}{\pi}x^2 + \frac{3}{\pi^2}x^3 - \tan x\right)\Big|_0^{\pi/3}$$
$$= \pi\left(\frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{9} - \sqrt{3}\right) = \frac{7\pi^2}{9} - \sqrt{3}\pi.$$



44. $x = 2, \quad x = 3, \quad y = 16 - x^4, \quad y = 0,$ about y-axis

SOLUTION Rotating the region enclosed by $x = 2$, $x = 3$, $y = 16 - x^4$ and the x-axis (shown in the figure below) about the y-axis produces a solid whose cross sections are washers with outer radius $R = 3$ and inner radius $r = \sqrt[4]{16 - y}$. The volume of the solid of revolution is

$$\pi\int_{-65}^{0} \left(9 - \sqrt{16 - y}\right) dy = \left(9y + \frac{2}{3}(16 - y)^{3/2}\right)\Big|_{-65}^{0} = \frac{425\pi}{3}.$$



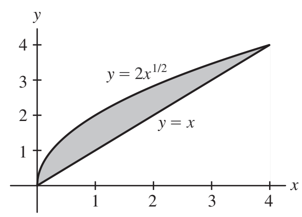
45. $y = 2\sqrt{x}$, $y = x$, about $x = -2$

SOLUTION Setting $2\sqrt{x} = x$ and squaring both sides yields

$$4x = x^2 \quad \text{or} \quad x(x-4) = 0,$$

so the two curves intersect at $x = 0$ and $x = 4$. Rotating the region enclosed by $y = 2\sqrt{x}$ and $y = x$ (see the figure below) about $x = -2$ produces a solid whose cross sections are washers with outer radius $R = y - (-2) = y + 2$ and inner radius $r = \frac{1}{4}y^2 - (-2) = \frac{1}{4}y^2 + 2$. The volume of the solid of revolution is

$$\begin{aligned} V &= \pi \int_0^4 \left((y+2)^2 - \left(\frac{1}{4}y^2 + 2 \right)^2 \right) dy \\ &= \pi \int_0^4 \left(4y - \frac{1}{16}y^4 \right) dy \\ &= \pi \left(2y^2 - \frac{1}{80}y^5 \right)\Big|_0^4 \\ &= \pi \left(32 - \frac{64}{5} \right) = \frac{96\pi}{5}. \end{aligned}$$



46. $y = 2\sqrt{x}$, $y = x$, about $y = 4$

SOLUTION Setting $2\sqrt{x} = x$ and squaring both sides yields

$$4x = x^2 \quad \text{or} \quad x(x-4) = 0,$$

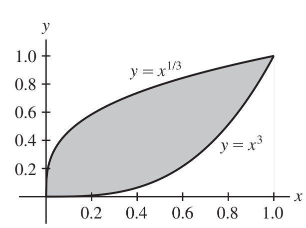
so the two curves intersect at $x = 0$ and $x = 4$. Rotating the region enclosed by $y = 2\sqrt{x}$ and $y = x$ (see the figure from the previous exercise) about $y = 4$ produces a solid whose cross sections are washers with outer radius $R = 4 - x$ and inner radius $r = 4 - 2\sqrt{x}$. The volume of the solid of revolution is

$$\begin{aligned} V &= \pi \int_0^4 \left((4-x)^2 - (4 - 2\sqrt{x})^2 \right) dy \\ &= \pi \int_0^4 \left(x^2 - 12x + 16\sqrt{x} \right) dy \\ &= \pi \left(\frac{1}{3}x^3 - 6x^2 + \frac{32}{3}x^{3/2} \right)\Big|_0^4 \\ &= \pi \left(\frac{64}{3} - 96 + \frac{256}{3} \right) = \frac{32\pi}{3}. \end{aligned}$$

47. $y = x^3$, $y = x^{1/3}$, for $x \geq 0$, about y-axis

SOLUTION Rotating the region enclosed by $y = x^3$ and $y = x^{1/3}$ (shown in the figure below) about the y-axis produces a solid whose cross sections are washers with outer radius $R = y^{1/3}$ and inner radius $r = y^3$. The volume of the solid of revolution is

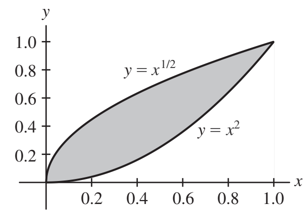
$$\pi \int_0^1 \left((y^{1/3})^2 - (y^3)^2 \right) dy = \pi \int_0^1 (y^{2/3} - y^6)\, dy = \pi \left(\frac{3}{5}y^{5/3} - \frac{1}{7}y^7 \right)\Big|_0^1 = \frac{16\pi}{35}.$$



48. $y = x^2, \quad y = x^{1/2}, \quad \text{about } x = -2$

SOLUTION Rotating the region enclosed by $y = x^2$ and $y = x^{1/2}$ (shown in the figure below) about $x = -2$ produces a solid whose cross sections are washers with outer radius $R = \sqrt{y} - (-2) = \sqrt{y} + 2$ and inner radius $r = y^2 - (-2) = y^2 + 2$. The volume of the solid of revolution is

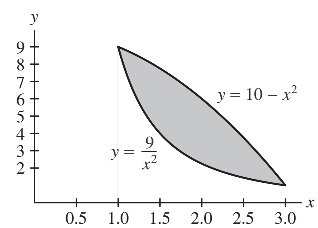
$$\pi \int_0^1 \left((\sqrt{y} + 2)^2 - (y^2 + 2)^2 \right) dy = \pi \int_0^1 \left(y + 4\sqrt{y} - y^4 - 4y^2 \right) dy$$
$$= \pi \left(\frac{1}{2}y^2 + \frac{8}{3}y^{3/2} - \frac{1}{5}y^5 - \frac{4}{3}y^3 \right) \Big|_0^1$$
$$= \pi \left(\frac{1}{2} + \frac{8}{3} - \frac{1}{5} - \frac{4}{3} \right) = \frac{49\pi}{30}.$$



49. $y = \dfrac{9}{x^2}, \quad y = 10 - x^2, \quad x \geq 0, \quad \text{about } y = 12$

SOLUTION The region enclosed by the two curves is shown in the figure below. Rotating this region about $y = 12$ produces a solid whose cross sections are washers with outer radius $R = 12 - 9x^{-2}$ and inner radius $r = 12 - (10 - x^2) = 2 + x^2$. The volume of the solid of revolution is

$$\pi \int_1^3 \left((12 - 9x^{-2})^2 - (x^2 + 2)^2 \right) dx = \pi \int_1^3 \left(140 - 4x^2 - x^4 - 216x^{-2} + 81x^{-4} \right) dx$$
$$= \pi \left(140x - \frac{4}{3}x^3 - \frac{1}{5}x^5 + 216x^{-1} - 27x^{-3} \right) \Big|_1^3 = \frac{1184\pi}{15}.$$



50. $y = \dfrac{9}{x^2}, \quad y = 10 - x^2, \quad x \geq 0, \quad \text{about } x = -1$

SOLUTION The region enclosed by the two curves is shown in the figure from the previous exercise. Rotating this region about $x = -1$ produces a solid whose cross sections are washers with outer radius $R = \sqrt{10 - y} - (-1) = \sqrt{10 - y} + 1$ and inner radius $r = 3y^{-1/2} - (-1) = 3y^{-1/2} + 1$. The volume of the solid of revolution is

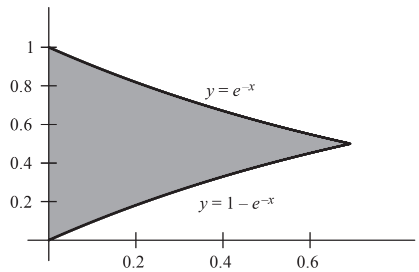
$$V = \pi \int_1^9 \left((\sqrt{10 - y} + 1)^2 - (3y^{-1/2} + 1)^2 \right) dy$$
$$= \pi \int_1^9 \left(10 - y + 2\sqrt{10 - y} - 9y^{-1} - 6y^{-1/2} \right) dy$$
$$= \pi \left(10y - \frac{1}{2}y^2 - \frac{4}{3}(10 - y)^{3/2} - 9 \ln y - 12\sqrt{y} \right) \Big|_1^9$$
$$= \pi \left(\left(90 - \frac{81}{2} - \frac{4}{3} - 9 \ln 9 - 36 \right) - \left(10 - \frac{1}{2} - 36 - 12 \right) \right)$$
$$= \pi \left(\frac{73}{6} - 9 \ln 9 + \frac{77}{2} \right) = \left(\frac{152}{3} - 9 \ln 9 \right) \pi.$$

51. $y = e^{-x}, \quad y = 1 - e^{-x}, \quad x = 0, \quad \text{about } y = 4$

SOLUTION Rotating the region enclosed by $y = 1 - e^{-x}$, $y = e^{-x}$ and the y-axis (shown in the figure below) about the line $y = 4$ produces a solid whose cross sections are washers with outer radius $R = 4 - (1 - e^{-x}) = 3 + e^{-x}$ and inner radius

$r = 4 - e^{-x}$. The volume of the solid of revolution is

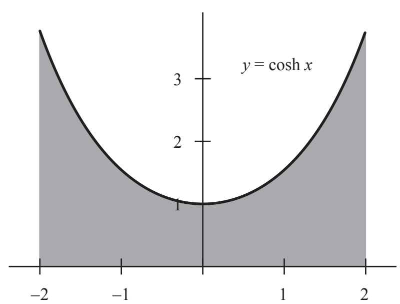
$$\pi \int_0^{\ln 2} \left((3 + e^{-x})^2 - (4 - e^{-x})^2 \right) dx = \pi \int_0^{\ln 2} (14e^{-x} - 7)\, dx = \pi(-14e^{-x} - 7x)\Big|_0^{\ln 2}$$
$$= \pi(-7 - 7\ln 2 + 14) = 7\pi(1 - \ln 2).$$



52. $y = \cosh x, \quad x = \pm 2, \quad$ about x-axis

SOLUTION Rotating the region enclosed by $y = \cosh x$, $x = \pm 2$ and the x-axis (shown in the figure below) about the x-axis produces a solid whose cross sections are disks with radius $R = \cosh x$. The volume of the solid of revolution is

$$\pi \int_{-2}^{2} \cosh^2 x\, dx = \frac{1}{2}\pi \int_{-2}^{2} (1 + \cosh 2x)\, dx = \frac{1}{2}\pi \left(x + \frac{1}{2}\sinh 2x \right) \Big|_{-2}^{2}$$
$$= \frac{1}{2}\pi \left[\left(2 + \frac{1}{2}\sinh 4 \right) - \left(-2 + \frac{1}{2}\sinh(-4) \right) \right] = \frac{1}{2}\pi(4 + \sinh 4).$$



53. The bowl in Figure 4(A) is 21 cm high, obtained by rotating the curve in Figure 4(B) as indicated. Estimate the volume capacity of the bowl shown by taking the average of right- and left-endpoint approximations to the integral with $N = 7$. The inner radii (in cm) starting from the top are 0, 4, 7, 8, 10, 13, 14, 20.

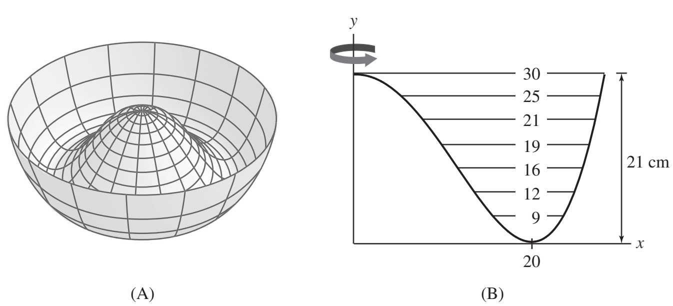


FIGURE 4

SOLUTION Using the given values for the inner radii and the values in Figure 4(B), which indicate the difference between the inner and outer radii, we find

$$R_7 = 3\pi \left((23^2 - 14^2) + (25^2 - 13^2) + (26^2 - 10^2) + (27^2 - 8^2) + (28^2 - 7^2) + (29^2 - 4^2) + (30^2 - 0^2) \right)$$
$$= 3\pi(4490) = 13470\pi$$

and

$$L_7 = 3\pi \left((20^2 - 20^2) + (23^2 - 14^2) + (25^2 - 13^2) + (26^2 - 10^2) + (27^2 - 8^2) + (28^2 - 7^2) + (29^2 - 4^2) \right)$$
$$= 3\pi(3590) = 10770\pi$$

Averaging these two values, we estimate that the volume capacity of the bowl is

$$V = 12120\pi \approx 38076.1 \text{ cm}^3.$$

54. The region between the graphs of $f(x)$ and $g(x)$ over $[0, 1]$ is revolved about the line $y = -3$. Use the midpoint approximation with values from the following table to estimate the volume V of the resulting solid.

x	0.1	0.3	0.5	0.7	0.9
$f(x)$	8	7	6	7	8
$g(x)$	2	3.5	4	3.5	2

SOLUTION The volume of the resulting solid is

$$\begin{aligned} V &= \pi \int_0^1 \left((f(x) + 3)^2 - (g(x) + 3)^2 \right) dx \\ &\approx 0.2\pi \left((11^2 - 5^2) + (10^2 - 6.5^2) + (9^2 - 7^2) + (10^2 - 6.5^2) + (11^2 - 5^2) \right) \\ &= 0.2\pi(96 + 57.75 + 32 + 57.75 + 96) = 67.9\pi. \end{aligned}$$

55. Find the volume of the cone obtained by rotating the region under the segment joining $(0, h)$ and $(r, 0)$ about the y-axis.

SOLUTION The segment joining $(0, h)$ and $(r, 0)$ has the equation

$$y = -\frac{h}{r}x + h \quad \text{or} \quad x = \frac{r}{h}(h - y).$$

Rotating the region under this segment about the y-axis produces a cone with volume

$$\begin{aligned} \frac{\pi r^2}{h^2} \int_0^h (h - y)^2 \, dx &= -\frac{\pi r^2}{3h^2}(h - y)^3 \Big|_0^h \\ &= \frac{1}{3}\pi r^2 h. \end{aligned}$$

56. The **torus** (doughnut-shaped solid) in Figure 5 is obtained by rotating the circle $(x - a)^2 + y^2 = b^2$ around the y-axis (assume that $a > b$). Show that it has volume $2\pi^2 ab^2$. *Hint:* Evaluate the integral by interpreting it as the area of a circle.

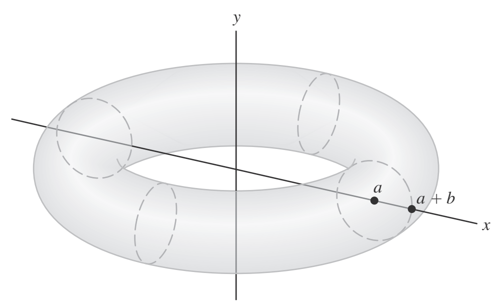


FIGURE 5 Torus obtained by rotating a circle about the y-axis.

SOLUTION Rotating the region enclosed by the circle $(x - a)^2 + y^2 = b^2$ about the y-axis produces a torus whose cross sections are washers with outer radius $R = a + \sqrt{b^2 - y^2}$ and inner radius $r = a - \sqrt{b^2 - y^2}$. The volume of the torus is then

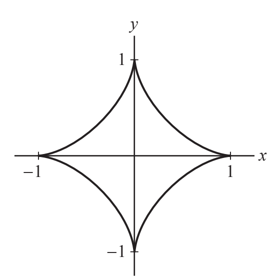
$$\pi \int_{-b}^{b} \left(\left(a + \sqrt{b^2 - y^2} \right)^2 - \left(a - \sqrt{b^2 - y^2} \right)^2 \right) dy = 4a\pi \int_{-b}^{b} \sqrt{b^2 - y^2} \, dy.$$

Now, the remaining definite integral is one-half the area of a circle of radius b; therefore, the volume of the torus is

$$4a\pi \cdot \frac{1}{2}\pi b^2 = 2\pi^2 ab^2.$$

57. GU Sketch the hypocycloid $x^{2/3} + y^{2/3} = 1$ and find the volume of the solid obtained by revolving it about the x-axis.

SOLUTION A sketch of the hypocycloid is shown below.



For the hypocycloid, $y = \pm\left(1 - x^{2/3}\right)^{3/2}$. Rotating this region about the x-axis will produce a solid whose cross sections are disks with radius $R = \left(1 - x^{2/3}\right)^{3/2}$. Thus the volume of the solid of revolution will be

$$\pi \int_{-1}^{1} \left((1 - x^{2/3})^{3/2}\right)^2 dx = \pi \left(\frac{-x^3}{3} + \frac{9}{7}x^{7/3} - \frac{9}{5}x^{5/3} + x \right)\Bigg|_{-1}^{1} = \frac{32\pi}{105}.$$

58. The solid generated by rotating the region between the branches of the hyperbola $y^2 - x^2 = 1$ about the x-axis is called a **hyperboloid** (Figure 6). Find the volume of the hyperboloid for $-a \le x \le a$.

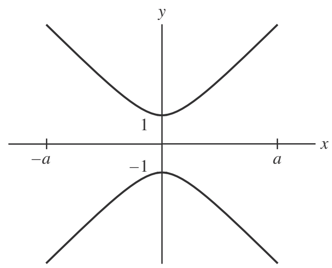


FIGURE 6 The hyperbola with equation $y^2 - x^2 = 1$.

SOLUTION Each cross section is a disk of radius $R = \sqrt{1 + x^2}$, so the volume of the hyperboloid is

$$\pi \int_{-a}^{a} \left(\sqrt{1 + x^2}\right)^2 dx = \pi \int_{-a}^{a} (1 + x^2)\, dx = \pi \left(x + \frac{1}{3}x^3 \right)\Bigg|_{-a}^{a} = \pi \left(\frac{2a^3 + 6a}{3} \right)$$

59. A "bead" is formed by removing a cylinder of radius r from the center of a sphere of radius R (Figure 7). Find the volume of the bead with $r = 1$ and $R = 2$.

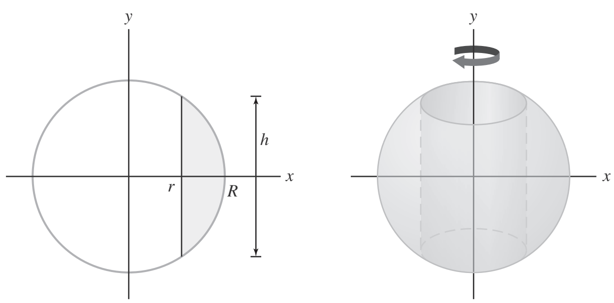


FIGURE 7 A bead is a sphere with a cylinder removed.

SOLUTION The equation of the outer circle is $x^2 + y^2 = 2^2$, and the inner cylinder intersects the sphere when $y = \pm\sqrt{3}$. Each cross section of the bead is a washer with outer radius $\sqrt{4 - y^2}$ and inner radius 1, so the volume is given by

$$\pi \int_{-\sqrt{3}}^{\sqrt{3}} \left(\left(\sqrt{4 - y^2}\right)^2 - 1^2 \right) dy = \pi \int_{-\sqrt{3}}^{\sqrt{3}} \left(3 - y^2\right) dy = 4\pi\sqrt{3}.$$

Further Insights and Challenges

60. Find the volume V of the bead (Figure 7) in terms of r and R. Then show that $V = \frac{\pi}{6}h^3$, where h is the height of the bead. This formula has a surprising consequence: Since V can be expressed in terms of h alone, it follows that two beads of height 1 cm, one formed from a sphere the size of an orange and the other from a sphere the size of the earth, would have the same volume! Can you explain intuitively how this is possible?

SOLUTION The equation for the outer circle of the bead is $x^2 + y^2 = R^2$, and the inner cylinder intersects the sphere when $y = \pm\sqrt{R^2 - r^2}$. Each cross section of the bead is a washer with outer radius $\sqrt{R^2 - y^2}$ and inner radius r, so the volume is

$$\pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \left(\left(\sqrt{R^2 - y^2}\right)^2 - r^2 \right) dy = \pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} (R^2 - r^2 - y^2)\, dy$$

$$= \pi \left((R^2 - r^2)y - \frac{1}{3}y^3 \right)\Bigg|_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} = \frac{4}{3}(R^2 - r^2)^{3/2}\pi.$$

Now, $h = 2\sqrt{R^2 - r^2} = 2(R^2 - r^2)^{1/2}$, which gives $h^3 = 8(R^2 - r^2)^{3/2}$ and finally $(R^2 - r^2)^{3/2} = \frac{1}{8}h^3$. Substituting into the expression for the volume gives $V = \frac{\pi}{6}h^3$. The beads may have the same volume but clearly the wall of the earth-sized bead must be extremely thin while the orange-sized bead would be thicker.

61. The solid generated by rotating the region inside the ellipse with equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ around the x-axis is called an **ellipsoid**. Show that the ellipsoid has volume $\frac{4}{3}\pi ab^2$. What is the volume if the ellipse is rotated around the y-axis?

SOLUTION

- Rotating the ellipse about the x-axis produces an ellipsoid whose cross sections are disks with radius $R = b\sqrt{1-(x/a)^2}$. The volume of the ellipsoid is then

$$\pi \int_{-a}^{a} \left(b\sqrt{1-(x/a)^2}\right)^2 dx = b^2\pi \int_{-a}^{a} \left(1 - \frac{1}{a^2}x^2\right) dx = b^2\pi \left(x - \frac{1}{3a^2}x^3\right)\Big|_{-a}^{a} = \frac{4}{3}\pi ab^2.$$

- Rotating the ellipse about the y-axis produces an ellipsoid whose cross sections are disks with radius $R = a\sqrt{1-(y/b)^2}$. The volume of the ellipsoid is then

$$\int_{-b}^{b} \left(a\sqrt{1-(y/b)^2}\right)^2 dy = a^2\pi \int_{-b}^{b} \left(1 - \frac{1}{b^2}y^2\right) dy = a^2\pi \left(y - \frac{1}{3b^2}y^3\right)\Big|_{-b}^{b} = \frac{4}{3}\pi a^2 b.$$

62. The curve $y = f(x)$ in Figure 8, called a **tractrix**, has the following property: the tangent line at each point (x, y) on the curve has slope

$$\frac{dy}{dx} = \frac{-y}{\sqrt{1-y^2}}$$

Let R be the shaded region under the graph of $0 \le x \le a$ in Figure 8. Compute the volume V of the solid obtained by revolving R around the x-axis in terms of the constant $c = f(a)$. *Hint:* Use the substitution $u = f(x)$ to show that

$$V = \pi \int_c^1 u\sqrt{1-u^2}\, du$$

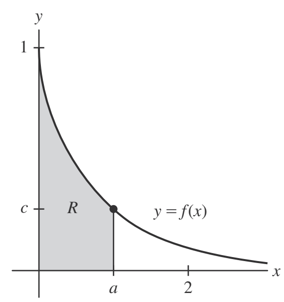


FIGURE 8 The tractrix.

SOLUTION Let $y = f(x)$ be the tractrix depicted in Figure 8. Rotating the region R about the x-axis produces a solid whose cross sections are disks with radius $f(x)$. The volume of the resulting solid is then

$$V = \pi \int_0^a [f(x)]^2\, dx.$$

Now, let $u = f(x)$. Then

$$du = f'(x)\, dx = \frac{-f(x)}{\sqrt{1-[f(x)]^2}}\, dx = \frac{-u}{\sqrt{1-u^2}}\, dx;$$

hence,

$$dx = -\frac{\sqrt{1-u^2}}{u}\, du,$$

and

$$V = \pi \int_1^c u^2 \left(-\frac{\sqrt{1-u^2}}{u}\, du\right) = \pi \int_c^1 u\sqrt{1-u^2}\, du.$$

Carrying out the integration, we find

$$V = -\frac{\pi}{3}(1-u^2)^{3/2}\Big|_c^1 = \frac{\pi}{3}(1-c^2)^{3/2}.$$

63. Verify the formula

$$\int_{x_1}^{x_2} (x - x_1)(x - x_2)\, dx = \frac{1}{6}(x_1 - x_2)^3 \qquad \boxed{3}$$

Then prove that the solid obtained by rotating the shaded region in Figure 9 about the x-axis has volume $V = \frac{\pi}{6}BH^2$, with B and H as in the figure. *Hint:* Let x_1 and x_2 be the roots of $f(x) = ax + b - (mx + c)^2$, where $x_1 < x_2$. Show that

$$V = \pi \int_{x_1}^{x_2} f(x)\, dx$$

and use Eq. (3).

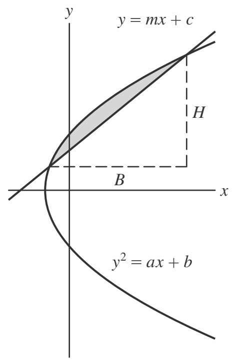


FIGURE 9 The line $y = mx + c$ intersects the parabola $y^2 = ax + b$ at two points above the x-axis.

SOLUTION First, we calculate

$$\int_{x_1}^{x_2} (x - x_1)(x - x_2)\, dx = \left(\frac{1}{3}x^3 - \frac{1}{2}(x_1 + x_2)x^2 + x_1x_2x\right)\Bigg|_{x_1}^{x_2} = \frac{1}{6}x_1^3 - \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_1x_2^2 - \frac{1}{6}x_2^3$$

$$= \frac{1}{6}\left(x_1^3 - 3x_1^2x_2 + 3x_1x_2^2 - x_2^3\right) = \frac{1}{6}(x_1 - x_2)^3.$$

Now, consider the region enclosed by the parabola $y^2 = ax + b$ and the line $y = mx + c$, and let x_1 and x_2 denote the x-coordinates of the points of intersection between the two curves with $x_1 < x_2$. Rotating the region about the y-axis produces a solid whose cross sections are washers with outer radius $R = \sqrt{ax + b}$ and inner radius $r = mx + c$. The volume of the solid of revolution is then

$$V = \pi \int_{x_1}^{x_2} \left(ax + b - (mx + c)^2\right) dx$$

Because x_1 and x_2 are roots of the equation $ax + b - (mx + c)^2 = 0$ and $ax + b - (mx + c)^2$ is a quadratic polynomial in x with leading coefficient $-m^2$, it follows that $ax + b - (mx + c)^2 = -m^2(x - x_1)(x - x_2)$. Therefore,

$$V = -\pi m^2 \int_{x_1}^{x_2} (x - x_1)(x - x_2)\, dx = \frac{\pi}{6}m^2(x_2 - x_1)^3,$$

where we have used Eq. (3). From the diagram, we see that

$$B = x_2 - x_1 \qquad \text{and} \qquad H = mB,$$

so

$$V = \frac{\pi}{6}m^2B^3 = \frac{\pi}{6}B\,(mB)^2 = \frac{\pi}{6}BH^2.$$

64. Let R be the region in the unit circle lying above the cut with the line $y = mx + b$ (Figure 10). Assume the points where the line intersects the circle lie above the x-axis. Use the method of Exercise 63 to show that the solid obtained by rotating R about the x-axis has volume $V = \frac{\pi}{6}hd^2$, with h and d as in the figure.

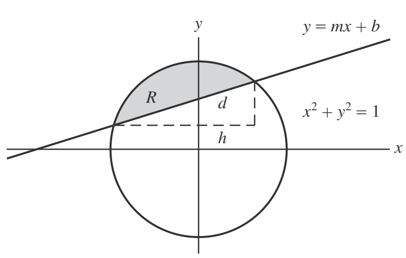


FIGURE 10

SOLUTION Let x_1 and x_2 denote the x-coordinates of the points of intersection between the circle $x^2 + y^2 = 1$ and the line $y = mx + b$ with $x_1 < x_2$. Rotating the region enclosed by the two curves about the x-axis produces a solid whose cross sections are washers with outer radius $R = \sqrt{1 - x^2}$ and inner radius $r = mx + b$. The volume of the resulting solid is then

$$V = \pi \int_{x_1}^{x_2} \left((1 - x^2) - (mx + b)^2\right) dx$$

Because x_1 and x_2 are roots of the equation $(1 - x^2) - (mx + b)^2 = 0$ and $(1 - x^2) - (mx + b)^2$ is a quadratic polynomial in x with leading coefficient $-(1 + m^2)$, it follows that $(1 - x^2) - (mx + b)^2 = -(1 + m^2)(x - x_1)(x - x_2)$. Therefore,

$$V = -\pi(1 + m^2) \int_{x_1}^{x_2} (x - x_1)(x - x_2)\, dx = \frac{\pi}{6}(1 + m^2)(x_2 - x_1)^3.$$

From the diagram, we see that $h = x_2 - x_1$. Moreover, by the Pythagorean theorem, $d^2 = h^2 + (mh)^2 = (1 + m^2)h^2$. Thus,

$$V = \frac{\pi}{6}(1 + m^2)h^3 = \frac{\pi}{6}h\left[(1 + m^2)h^2\right] = \frac{\pi}{6}hd^2.$$

6.4 The Method of Cylindrical Shells

Preliminary Questions

1. Consider the region $\mathcal{R}$ under the graph of the constant function $f(x) = h$ over the interval $[0, r]$. Give the height and the radius of the cylinder generated when $\mathcal{R}$ is rotated about:

(a) the x-axis **(b)** the y-axis

SOLUTION

(a) When the region is rotated about the x-axis, each shell will have radius h and height r.

(b) When the region is rotated about the y-axis, each shell will have radius r and height h.

2. Let V be the volume of a solid of revolution about the y-axis.

(a) Does the Shell Method for computing V lead to an integral with respect to x or y?

(b) Does the Disk or Washer Method for computing V lead to an integral with respect to x or y?

SOLUTION

(a) The Shell method requires slicing the solid parallel to the axis of rotation. In this case, that will mean slicing the solid in the vertical direction, so integration will be with respect to x.

(b) The Disk or Washer method requires slicing the solid perpendicular to the axis of rotation. In this case, that means slicing the solid in the horizontal direction, so integration will be with respect to y.

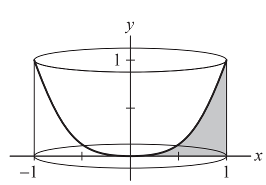
Exercises

In Exercises 1–6, sketch the solid obtained by rotating the region underneath the graph of the function over the given interval about the y-axis, and find its volume.

1. $f(x) = x^3$, $[0, 1]$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height x^3, so the volume of the solid is

$$2\pi \int_0^1 x \cdot x^3\, dx = 2\pi \int_0^1 x^4\, dx = 2\pi \left(\frac{1}{5}x^5\right)\Big|_0^1 = \frac{2}{5}\pi.$$



2. $f(x) = \sqrt{x}$, $[0, 4]$

SOLUTION A sketch of the solid is shown below. Each shell has radius x and height $\sqrt{x}$, so the volume of the solid is

$$2\pi \int_0^4 x\sqrt{x}\, dx = 2\pi \int_0^4 x^{3/2}\, dx = 2\pi \left(\frac{2}{5}x^{5/2}\right)\Big|_0^4 = \frac{128}{5}\pi.$$