DUSO Mathematics League 2011 - 2012

CONTEST #2.

SOLUTIONS

2 - 1. We subtract $\frac{n+2}{n+3} - \frac{n}{n+1} = \frac{2}{n^2+4n+3} = \frac{1}{40} = \frac{2}{80}$, so $n^2+4n+3=80$ and $n^2+4n-77=0$. This factors into $(n+11)(n-7)=0$, so $n=-11$ or $n=7$. We reject the negative value of n, so the fraction we want is $\mathbf{\frac{7}{8}}$.

2 - 2. Since each of the first three fractions is equal to 2, we have $a=2b+8$, $b=2c+12$, and $c=2a+16$. Adding the three equations together, we obtain $a+b+c=2(a+b+c)+36$, so $a+b+c=\mathbf{-36}$.

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2 - 3. Any two sides of a triangle must add up to be bigger than the third side. Thus, the third side must measure more than $20-11=9$, but less than $20+11=31$. We count up the number of integers in the set $\{10,11,12,\cdots,30\}$ to obtain **21**.

2 - 4. The Pythagorean Theorem, applied to $\triangle ACB$, tells us that $AB=10$ and $MB=5$. Note that $\triangle ACB$ is similar to $\triangle DMB$, so that $\frac{DM}{5}=\frac{6}{8}$, so $DM=\frac{30}{8}=\frac{15}{4}$. Thus, the area of $\triangle DMB$ is $\frac{1}{2}\cdot\frac{15}{4}\cdot 5=\mathbf{\frac{75}{8}}$.

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2 - 5. Let the two-digit number be $\underline{TU}=10T+U$. We know that $10T+U=3(T+U)$, and we collect like terms to see that $7T=2U$. Now we look for possible solutions. If $T=1$, U is not a whole number. If $T=2$, then $U=7$. If $T\geq 3$, then U cannot be a single digit. Thus, there is only one such number: **27**.

2 - 6. We use the following formula for area of a triangle: $[BEN]=\frac{1}{2}be\sin N$. Supposing that the two given sides are b and e, we obtain $10\sqrt{3}=\frac{1}{2}\cdot 5\cdot 7\cdot\sin N$. This means that $\sin N=\frac{20\sqrt{3}}{35}$. Since $\sin^2 N+\cos^2 N=1$, we obtain $\cos N=-\frac{1}{7}$ (note that $\cos N$ is negative since the triangle is obtuse). Now, we use the Law of Cosines: $n^2=5^2+7^2-2\cdot 5\cdot 7\cdot -\frac{1}{7}\Rightarrow n^2=84\Rightarrow n=\mathbf{2\sqrt{21}}$.

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T-1. Ryan, Sam, and Tyler are playing a 3-player game. In each round, the winner gets 12 points, the second-place player gets 5 points, and the third-place player gets 0 points. After ten rounds of play, Ryan has 75 points and Sam has 34 points. Compute the number of times Tyler finished second.

T-1Sol. $\boxed{5}$ The total number of points scored over the 10 rounds is $17 \cdot 10 = 170$ points. After 10 rounds, Tyler has $170 - 75 - 34 = 61$ points. This comes from adding a multiple of 5 to a multiple of 12. We see that Tyler has 3 first place finishes and 5 second place finishes. The number we want is **5**.

T-2. If two of the roots of $x^6 - 5x^5 - 34x^4 + 242x^3 - 572x^2 + 1048x - 1680 = 0$ are $2i$ and $3+i$, compute the real roots.

T-2Sol. $\boxed{\{-7, 6\}}$ Since all of the coefficients of the polynomial are integers, complex roots must come in conjugal pairs. Since $2i$ is a root, $-2i$ must also be a root, and since $3+i$ is a root, so is $3-i$. Thus, the sum of the six roots is $2i + (-2i) + (3+i) + (3-i) + r_5 + r_6 = -\frac{-5}{1}$, so $r_5 + r_6 = -1$. The product of the six roots is $2i \cdot (-2i) \cdot (3+i) \cdot (3-i) \cdot r_5 \cdot r_6 = \frac{-1680}{1}$, so $4 \cdot 10 \cdot r_5 \cdot r_6 = -1680$, and therefore $r_5 \cdot r_6 = -42$. We may solve this system to obtain the real roots: $\{\mathbf{-7}, \mathbf{6}\}$.

T-3. At Springfield High School, there are 224 seniors. Some own only a car, some own only a bike, and some own a car and a bike. Every senior owns at least a car or a bike. The ratio of the number of students who own only a car to the number of students who own only a bike is $3:7$. The ratio of the number of students who own a car to the number of students who own a bike is $7:11$. If a senior is chosen at random, compute the probability that the student owns a car and a bike.

T-3Sol. $\boxed{\frac{2}{7}}$ Let the number of students who own only a car be given by $3x$. Then the number of students who own only a bike is $7x$. Let the number of students who own both a car and a bike be y. Then we have $\frac{3x+y}{7x+y} = \frac{7}{11}$. After cross-multiplication, we obtain $33x + 11y = 49x + 7y$, so $4y = 16x$ and $y = 4x$. Since the whole population of the senior class is 224, we solve $3x + 4x + 7x = 224$ to obtain $x = 16$ and $y = 4 \cdot 16 = 64$. The probability we want is $\frac{64}{224} = \frac{2}{7}$.

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